Lorentz Group

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1 Lorentz group

In the derivation of Dirac equation it is not clear what is the meaning of the Dirac γ matrices. It turns out that they are related to representations of Lorentz group. The Lorentz group is a collection of linear transformations of space-time coordinates

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$$

which leaves the proper time

$$\tau^2 = (x^o)^2 - (\overrightarrow{x})^2 = x^\mu x^\nu g_{\mu\nu} = x^2$$

invariant. This requires the transformation matrix Λ^μ_ν to satisfy the pseudo-orthogonality relation,

$$\Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta} g_{\mu\nu} = g_{\alpha\beta}$$

1.1 Generators

It is useful to investigate the group structure by studying their infinitesmal elements near the identity, the generators. For infinitesmal transformation, we write

$$\Lambda^\mu_{\ \alpha} = g^\mu_\alpha + \epsilon^\mu_{\ \alpha} \qquad \text{with} \quad |\varepsilon^\mu_\alpha| \ll 1$$

As before, the pseudo-orthogonality relation implies

$$(g^\mu_\alpha + \epsilon^\mu_{\ \alpha})\,(g^\nu_\beta + \epsilon^\nu_{\ \beta}) g_{\mu\nu} = g_{\alpha\beta}$$

Or

$$\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0$$

We see that $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. So there are 6 independent group parameters.

Example: Boost along $x - axis$

$$\Lambda^\mu_{\ \alpha} = \begin{pmatrix} \cosh\omega & \sinh\omega & 0 & 0 \\ \sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This implies that

$$\varepsilon^0_{\ 1} = \omega, \qquad \varepsilon^1_{\ 0} = \omega \tag{1}$$

Or

$$\varepsilon_{01} = \omega, \qquad \varepsilon_{10} = -\omega$$

Similarly, ε_{02}, and ε_{03} correspond to boosts in y and z directions respectively.

Example: rotation about $z-axis$

$$\Lambda^{\mu}_{\ \alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$\varepsilon^{1}_{\ 2} = -\theta, \qquad \varepsilon^{2}_{\ 1} = \theta$$

Or

$$\varepsilon_{12} = \theta, \qquad \varepsilon_{21} = -\theta$$

Similarly, ε_{23},and ε_{31} correspond to rotations about $x-$ and y axis respectively.

Consider $f(x^{\mu})$, an arbitrary function of x^{μ}. Under the infinitesimal Lorentz transformation, the change in f is

$$\begin{aligned} f(x^{\mu}) &\rightarrow f(x'^{\mu}) = f(x^{\mu} + \varepsilon^{\mu}_{\alpha} x^{\alpha}) \approx f(x^{\mu}) + \varepsilon_{\alpha\beta} x^{\beta} \partial_{\alpha} f + \cdots \\ &= f(x^{\mu}) + \frac{1}{2}\varepsilon_{\alpha\beta}[x^{\beta}\partial^{\alpha} - x^{\alpha}\partial^{\beta}] f(x) + \cdots \end{aligned}$$

Introduce operators $M_{\mu\nu}$ to represent this change,

$$f(x') = f(x) - \frac{i}{2}\varepsilon_{\alpha\beta} M^{\alpha\beta} f(x) + \cdots$$

then

$$M^{\alpha\beta} = -i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}) \tag{2}$$

generators $M_{\mu\nu}$ are called the generators of Lorentz group operating on functions of space-time coordinates. Note that for $\alpha, \beta = 1, 2, 3$ these are just the angular momentum operator $L_{ij} = i\,(x_i\partial_j - x_j\partial_i)$.

Using the generators given in Eq(2) it is straightforward to work out commutators of these generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\delta}M_{\beta\gamma})$$

Define

$$M_{ij} = \epsilon_{ijk} J_k, \quad M_{oi} = K_i$$

where $J_k' s$ correspond to the usual rotations and K_i the Lorentz boost operators. We can solve for J_i to get

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$$

The commutator of $J_i' s$ are,

$$[J_i, J_j] = \left(\frac{1}{2}\right)^2 \varepsilon_{ikl}\varepsilon_{jmn}[M_{kl}, M_{mn}] = (-i)\,(\frac{1}{2})^2 \varepsilon_{ikl}\varepsilon_{jmn}(g_{lm}M_{kn} - g_{km}M_{ln} - g_{ln}M_{km} + g_{kn}M_{lm})$$

$$= (\frac{1}{2})^2\,(-i)\,[-\epsilon_{ikl}\varepsilon_{jln}M_{kn} + \epsilon_{ikl}\varepsilon_{jkn}M_{ln} + \epsilon_{ikl}\varepsilon_{jml}M_{km} - \epsilon_{ikl}\varepsilon_{jmk}M_{lm}]$$

Using identity

$$\epsilon_{abc}\varepsilon_{alm} = (\delta_{bl}\delta_{cm} - \delta_{bm}\delta_{cl})$$

we get

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{3}$$

Thus we can identify J_i as the angular momentum operator.

Similarly, we can derive

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \qquad [J_i, K_j] = i\epsilon_{ijk}K_k \tag{4}$$

Eqs(3,4) are called the Lorentz algebra.

To simplify the Lorentz algebra, we define the combinations

$$A_i = \frac{1}{2}(J_i + iK_i) \quad , B_i = \frac{1}{2}(J_i - iK_i)$$

Then it is straightforward to derive the following commutation relations,

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0$$

This means that the algebra of Lorentz generators factorizes into 2 independent $SU(2)$ algebra. The representations are just the tensor products of the representation of $SU(2)$ algebra. We label the irreducible representation by (j_1, j_2) which transforms as $(2j_1+1)$-dim representation under A_i algebra and $(2j_2+1)$-dim representation under B_i algebra.

1.2 Simple representations

(a) $(\frac{1}{2}, 0)$ representation χ_a

This 2-component object has the following properties,

$$A_i\chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b \qquad \Longrightarrow \qquad \frac{1}{2}(J_i + iK_i)\chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b$$

$$B_i\chi_a = 0 \qquad \Longrightarrow \qquad \frac{1}{2}(J_i - iK_i)\chi_a = 0$$

Combining these realtions we get

$$\vec{J}\chi = (\frac{\vec{\sigma}}{2})\chi, \qquad \vec{K}\chi = -i(\frac{\vec{\sigma}}{2})\chi$$

(b) $(0, \frac{1}{2})$ representation η_a

Similarly, we can get

$$A_i\eta_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i)\eta_a = 0$$

$$B_i\eta_a = (\frac{\sigma_i}{2})_{ab} \qquad \Longrightarrow \quad \frac{1}{2}(J_i - iK_i)\eta_a = (\frac{\sigma_i}{2})_{ab}\eta_b$$

$$\vec{J}\eta = (\frac{\vec{\sigma}}{2})\eta, \qquad \vec{K}\eta = i(\frac{\vec{\sigma}}{2})\eta$$

If we define a 4-component ψ by putting togather these 2 representations,

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

Then the action of the Lorentz generators are

$$\vec{J}\psi = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix}\psi, \qquad \vec{K}\psi = \begin{pmatrix} -i\frac{\vec{\sigma}}{2} & 0 \\ 0 & i\frac{\vec{\sigma}}{2} \end{pmatrix}\psi \tag{5}$$

ψ are related to the 4-component Dirac field we studied before, but with different representation for the γ matrices. This can be seen as follows.
Consider Dirac matrices in the following form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{where} \quad \sigma^\mu = (1, \vec{\sigma}) \quad , \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

More explicitly,

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

It is straightforward to check that in this case.

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This means that in 4-component field $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$, χ is **right-handed** and η is **left-handed**. In this representation, usually called the **Weyl representaiton**, it is easy to check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\} = exp\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\}$$

Write $\varepsilon^{0i} = \beta^i$, $\varepsilon^{ij} = \varepsilon^{ijk}\theta^k$

$$\sigma_{ij}\epsilon^{ij} = \varepsilon^{ijk}\theta^k\epsilon_{ijl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = 2\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} \end{pmatrix}$$

$$\sigma_0 i\varepsilon^0 i = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

$\Rightarrow$

$$-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij}) = \frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

More precisely,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\}\psi = \exp\left[\frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}\right]\psi \tag{6}$$

If we write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$

then in terms of the generators $\vec{J}$, $\vec{K}$

$$L = \exp\left[(-i)\left(\vec{J}\cdot\vec{\theta} + \vec{K}\cdot\vec{\beta}\right)\right]$$

We then see from Eq(6) that for this ψ, $\vec{J}$, $\vec{K}$ are of the form,

$$\vec{J} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad \vec{K} = \frac{1}{2}\begin{pmatrix} -i\vec{\sigma} & 0 \\ 0 & i\vec{\sigma} \end{pmatrix}$$

These are the same as those in Eq(5). This demonstrate that the wavefunction which satisfies Dirac equation is just the representation $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ under the Lorentz group. Futhermore, the right-handed components transform as $\left(\frac{1}{2},0\right)$ represenation while left-handed components transform as $\left(0,\frac{1}{2}\right)$ representation.

Alternative choice is to use ψ_R and the complex conjugate ψ_R^* (sometime dotted indice are used for this basis) instead of ψ_R and ψ_L. Since

$$\vec{J}\psi_R = (\frac{\vec{\sigma}}{2})\psi_R, \qquad \vec{K}\psi_R = -i(\frac{\vec{\sigma}}{2})\psi_R$$

we get for the complex conjuate

$$\vec{J}\psi_R^* = (\frac{\vec{\sigma}^*}{2})\psi_R^*, \qquad \vec{K}\psi_R^* = i(\frac{\vec{\sigma}^*}{2})\psi_R^*$$

It is probably more clearer to use some other notation for ψ_R^*,

$$\vec{J}\chi = (\frac{\vec{\sigma}^*}{2})\chi, \qquad \vec{K}\chi = i(\frac{\vec{\sigma}^*}{2})\chi$$

Then

$$\vec{A}\chi = \frac{1}{2}(\vec{J} + i\vec{K})\chi = 0, \qquad \vec{B}\chi = \frac{1}{2}(\vec{J} - i\vec{K})\chi = (\frac{\vec{\sigma}^*}{2})\chi$$

and indeed χ belongs to the irrep $(0, \frac{1}{2})$.